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A SMOOTH CLOSED CURVE COMPOSED OF RECTILINEAR SEGMENTS WITH VERTEX POINTS WHICH ARE NOWHERE DENSE [*]

By E. R. Hedrick

It has occurred to the writer repeatedly that examples of an interesting nature result from simple integration of known continuous functions whose behavior is remarkable.[†] An instance of this kind is the example presented in this note.

A curve

$$(1) \qquad \begin{cases} x = f(t), \\ y = \phi(t), \end{cases} \qquad (t_0 \leqq t \leqq t_1)$$

is smooth, according to Professor Osgood [‡] if it has a continuously turning tangent at every point, that is, if $f(t)$ and $\phi(t)$ possess continuous first derivatives which satisfy the relation $f'(t)^2 + \phi'(t)^2 > 0$ for every value of t considered.

It is remarkable that there exist curves formed by joining together straight line segments, which are smooth in this sense. Such a curve evidently cannot have a corner, for either $f'(t)$ or $\phi'(t)$ would fail to exist at such a point. In order to grasp clearly the example which follows it is desirable to repeat here in some detail a well-known example.[§]

Let us remove from the interval $0 \leqq x \leqq 1$ first the interior points of the middle thirds, i, e., the points $1/3 < x < 2/3$; of the two segments of

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[*] Read at the summer meeting of the American Mathematical Society at Cornell University, Sept. 3, 1907.

[†] Another instance is given in a paper entitled "On a Function which occurs in the Law of the Mean," Annals of Mathematics, ser. 2, vol. 7, p. 177 (1906).

[‡] See Osgood, *Lehrbuch der Functionentheorie*, Leipzig, Teubner, 1906, vol. 1, p. 122.

[§] See, e. g., Osgood, l. c., p. 164 and 166; Lebesgue, *Leçons sur l'integration*, Paris, Gauthier-Villars, 1904, p. 13 and p. 26.

length 1/3 which remain let us remove the points interior to their middle third, i. e., the points $1/9 < x < 2/9$ and the points $7/9 < x < 8/9$; of the four segments of length 1/9 which remain let us remove the points interior to their middle thirds; and so on, the operation being carried on thus indefinitely. The points which remain after the whole operation is completed consist of the end-points of the intervals whose interiors have been removed, together with the limiting points of these end-points; we shall call this whole assemblage of points (E). The assemblage (E) is perfect and is nowhere dense in the interval $0 \leqq x \leqq 1$.

Let us now define a function $f(x)$ as follows: at a point $x = t$ which does not belong to (E), $f(t)$ shall be equal to the value of x at the middle point of the interval to which t belongs; at the points of (E), $f(x)$ is defined by the requirement that $f(x)$ shall be continuous, i. e., $\lim_{x=a} f(x) = f(a)$. This definition involves no contradiction; the function $f(x)$ defined is continuous in the interval $0 \leqq x \leqq 1$, it possesses a derivative and that derivative is zero except at the points of (E).* The figure is easy to construct; it consists of segments of straight lines parallel to the axis of x and of the same length as the intervals removed above, together with the limiting points of the end-points of these segments.†

Let us now form the integral of this function:

$$\phi(x) = \int_0^x f(x)\,dx; \tag{2}$$

the function $\phi(x)$ is defined for every value of x in the interval $0 \leqq x \leqq 1$, since $f(x)$ is everywhere continuous; moreover, the derivative of $\phi(x)$ exists at every point $0 \leqq x \leqq 1$, the derivatives at the points $x = 0$ and $x = 1$ being, of course, one-sided; and $\phi'(x) = f(x)$.

The curve

$$y = \phi(x) \tag{3}$$

is continuous and its derivative $\phi'(x)$ $[= f(x)]$ exists and is continuous at every point of the interval $0 \leqq x \leqq 1$. Since $f(x)$ is a constant in each of

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[*] The results quoted at this point and below are proved, for example, in the books just mentioned.

[†] It is important to visualize this figure; it is drawn, e. g., by Osgood, l. c.

the intervals removed to form (E), $\phi(x)$ is a linear function and the curve (3) is a straight line throughout each of those intervals. It is therefore clear that (3) represents a curve which is *smooth* in the sense above and which is composed of straight line segments except for the end-points of these line segments and the limiting points of these end-points; and *these exceptional points (vertices) are nowhere dense*, i. e., on any arc of the curve (3) there is at least one partial arc which is a straight line. Even at these exceptional points the curve is *smooth*. Nevertheless the whole curve (3) is not a straight line, for $\phi'(0) = f(0) = 0$ and $\phi'(1) = f(1) = 1$.

A combination of eight arcs, symmetrically arranged, similar to that defined by (3) evidently gives rise to a curve which is *closed*, *smooth* in Osgood's sense, and which is composed of rectilinear segments with the exception of a set of points which is not dense on any arc of the curve.

It is interesting to observe that (3) consists of the chords of the parabola $y = \frac{1}{2}x^2$ which connect those points of the parabola whose abscissae are the end-points of the intervals mentioned above, together with the limiting points of the end-points of these chords.

For, let $x = e$ be a point of (E); the area

$$\int_0^e f(x)dx = \phi(e)$$

may be thought of as made up of the areas under an arbitrary (but fixed) number of these rectilinear segments of $f(x)$ which lie above the interval $0 \leqq x \leqq e$, together with a residual area. The former area is obviously the same as the area under the corresponding portions of the curve $y = x$; the residual area can be made arbitrarily small, since the content of (E) is *nil*. Hence the total area is the same as that under $y = x$, or

$$\phi(e) = \int_0^e x\,dx = \tfrac{1}{2}e^2.$$

The same fact results at once from the consideration that the points of (E) are "of measure zero," whence they may be neglected in computing the integral.*

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* To show this, it is necessary to consider that the assemblage (E) is "of measure zero;" hence any or all of its points may be neglected in determining the integral. See, e. g., Lebesgue, l. c.

It follows that the closed curve mentioned is built up upon eight arcs of parabolas which fit together smoothly at their end-points. It would be equally simple, following this construction, to build up a closed curve of the same type composed of chords of a circle, together with the limiting points of the end-points of these chords. Had we proceeded in this manner, however, the method of reaching the result would have been disguised, and it would have been necessary to give an independent proof of the statements which we have been able to throw back upon the well-known results for the assemblage (E) and the function $f(x)$.

COLUMBIA, MO.,
JANUARY, 1908.